Notes on Consumer Theory

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*Most of these notes follow the third edition of the book "Advanced Microeconomic Theory" by G. Jehle & P. Reny. All errors are my own.

1 Primitive Notions

There are four building blocks in any model of consumer choice: the consumption set, the feasible set, the preference relation and the behavioral assumption. The combination of these blocks will build the perception of a given problem and how to approach it.

The **consumption set** can be interpreted as a set X that represents the alternatives, or complete consumption plans that a consumer can conceive; the consumption set is also called the choice set. Let $x_i \in \mathbb{R}$ represent the number of units of good i, where $\boldsymbol{x} = x_1, ..x_n$ is a **consumption bundle** or a consumption plan. It follows that a consumption bundle $\boldsymbol{x} \in X$ is represented by a point $\boldsymbol{x} \in \mathbb{R}^n$, where we usually can assume that the consumption is the positive orthant of the consumption set, $X = \mathbb{R}^n_+$.

Properties of the consumption set:

1. $X \subseteq \mathbb{R}^n_+$,
2. X is closed,[1]
3. X is convex,[2]
4. $\mathbf{0} \in X$

It follows that we need to define a **feasible set** as the set of consumption bundles that are conceivable and obtainable given a consumer's circumstances. Thus, we can say that the feasible set is defined as $B \subset X$.

A **preference relation** specifies the limits, if any, on the consumer's ability to specify her tastes for different objects of choice. Finally, the **behavioral assumption** specifies the guiding principle the consumer uses to make a choice. In general, we suppose that the consumer seeks to identify and select an available alternative that is most preferred according to her personal tastes.

2 Preferences and Utility

2.1 Preference Relations

Consumer preferences are characterized axiomatically, where these **axioms of consumer choice** give a formal mathematical expression to the fundamental aspects of consumer behavior and attitudes towards the objects of choice. Together, they formalize the belief that consumers can choose and that choices are consistent in a particular way.

We represent consumer preferences by the binary relation $\succeq$, defined on the consumption set, X. Whenever $\boldsymbol{x}^1 \succeq \boldsymbol{x}^2$, we say that $\boldsymbol{x}^1$ is at least as good as $\boldsymbol{x}^2$. The following axioms set forth basic criteria with which the preference relation must conform:

Axiom 2.1. ***Completeness.*** $\forall\, (\boldsymbol{x}^1, \boldsymbol{x}^2) \in X : \boldsymbol{x}^1 \succeq \boldsymbol{x}^2 \lor \boldsymbol{x}^2 \succeq \boldsymbol{x}^1$

Axiom 2.2. ***Transitivity.*** $\forall\, (\boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{x}^3) \in X : \boldsymbol{x}^1 \succeq \boldsymbol{x}^2 \land \boldsymbol{x}^2 \succeq \boldsymbol{x}^3 \implies \boldsymbol{x}^1 \succeq \boldsymbol{x}^3$

[1] S is a closed set if its complement S^c is an open set. S^c is open if $\forall\, \boldsymbol{x} \in S^c\ \exists\, \epsilon > 0 : B_\epsilon(\boldsymbol{x}) \subset S^c$. Where $B_\epsilon(\boldsymbol{x})$ denotes the open ball of radius ϵ centered at $\boldsymbol{x}$.

[2] $S \subset \mathbb{R}^n$ is a convex set if $\forall\, (\boldsymbol{x}_1, \boldsymbol{x}_2) \in S$ we have $t\boldsymbol{x}^1 + (1-t)\boldsymbol{x}^2 \in S$.

Axiom 1 formalizes the notion of the consumer being able to make choices, while Axiom 2 provides a formal notion for the requirement that the consumer choices have to be consistent. In other words that the consumer can rank any finite number of elements in the consumption set, X, from best to worse (without ruling out the possibility of ties).

Definition 2.1. The binary relation $\succeq$ on the consumption set X is called a **preference relation** if it satisfies Axioms 1 and 2.

Definition 2.2. The binary relation $\succ$ on the consumption set X is called a **strict preference relation**, and it is defined as:

$$\boldsymbol{x}^1 \succ \boldsymbol{x}^2 \iff \boldsymbol{x}^1 \succeq \boldsymbol{x}^2 \wedge \boldsymbol{x}^2 \not\succeq \boldsymbol{x}^1$$

Definition 2.3. The binary relation $\sim$ on the consumption set X is called an **indifference relation**, and it is defined as:

$$\boldsymbol{x}^1 \sim \boldsymbol{x}^2 \iff \boldsymbol{x}^1 \succeq \boldsymbol{x}^2 \wedge \boldsymbol{x}^2 \succeq \boldsymbol{x}^1$$

With these definitions we can finally establish that for any pair $(\boldsymbol{x}^1, \boldsymbol{x}^2)$, exactly one of three mutually exclusive possibilities holds: $\boldsymbol{x}^1 \succ \boldsymbol{x}^2$ or $\boldsymbol{x}^2 \succ \boldsymbol{x}^1$, or $\boldsymbol{x}^1 \sim \boldsymbol{x}^2$. Bearing this condition in mind, we can proceed to define some specific sets as a function of a single consumption alternative.

Definition 2.4. Let $\boldsymbol{x}^0$ be any point in the consumption set X. Relative to that point, we can define the following subsets of X:

1. $\succeq (\boldsymbol{x}^0) \equiv \{\boldsymbol{x} | \boldsymbol{x} \in X, \boldsymbol{x} \succeq \boldsymbol{x}^0\}$ Called "at least as good" set.
2. $\preceq (\boldsymbol{x}^0) \equiv \{\boldsymbol{x} | \boldsymbol{x} \in X, \boldsymbol{x}^0 \succeq \boldsymbol{x}\}$ Called "no better than" set.
3. $\succ (\boldsymbol{x}^0) \equiv \{\boldsymbol{x} | \boldsymbol{x} \in X, \boldsymbol{x} \succ \boldsymbol{x}^0\}$ Called "preferred to" set.
4. $\prec (\boldsymbol{x}^0) \equiv \{\boldsymbol{x} | \boldsymbol{x} \in X, \boldsymbol{x}^0 \succ \boldsymbol{x}\}$ Called "worse than" set.
5. $\sim (\boldsymbol{x}^0) \equiv \{\boldsymbol{x} | \boldsymbol{x} \in X, \boldsymbol{x} \sim \boldsymbol{x}^0\}$ Called "indifference" set.

Now we can continue to express regularity conditions on the preferences in order to avoid irregular behavior on choice making. From now on, we set $X = \mathbb{R}^n_+$:

Axiom 2.3. ***Continuity.*** *$\forall\, \boldsymbol{x} \in \mathbb{R}^n_+$, the "at least as good as" set , $\succeq (\boldsymbol{x}^0)$, and the "no better than" set, $\preceq (\boldsymbol{x}^0)$, are closed in $\mathbb{R}^n_+$.*

This axiom implies that if $\succeq (\boldsymbol{x}^0)$ is closed in $\mathbb{R}^n_+$, $\succ (\boldsymbol{x}^0)$ is open in in $\mathbb{R}^n_+$. Moreover, since $\succeq (\boldsymbol{x}^0)$ and $\preceq (\boldsymbol{x}^0)$ are closed, so is $\sim (\boldsymbol{x}^0)$. The continuity axiom guarantees that there are no sudden jumps on the consumption set.

To efficiently depict the view that a consumer may always choose something, or that there is some arrangement to a given bundle that she would prefer over another one, we can make use of another axiom. Denote $B_\epsilon(\boldsymbol{x}^0)$ as the open ball of radius ϵ centered at $\boldsymbol{x}^0$:

Axiom 2.4. ***Local non-satiation.*** $\forall\, \boldsymbol{x}^0 \in \mathbb{R}^n_+ \wedge \forall\, \epsilon > 0, \exists\, x \in B_\epsilon(\boldsymbol{x}^0) \cap \mathbb{R}^n_+ : \boldsymbol{x} \succ \boldsymbol{x}^0$

Local non-satiation requires a preferred nearby bundle to always exist, although it does not rule out the possibility of that bundle to involve less of one or even all commodities. We can add the assumption that a consumer always prefers more than less by including strict monotonicity. If the bundle $\boldsymbol{x}^0$ contains at least as much of every good as does $\boldsymbol{x}^1$, we write $\boldsymbol{x}^0 \geq \boldsymbol{x}^1$, while if it contains strictly more of every good than $\boldsymbol{x}^1$ we write $\boldsymbol{x}^0 >> \boldsymbol{x}^1$.

Axiom 2.5. ***Strict monotonicity.*** *$\forall\, \boldsymbol{x}^0, \boldsymbol{x}^1 \in \mathbb{R}^n_+$, if $\boldsymbol{x}^0 \geq \boldsymbol{x}^1$ then $\boldsymbol{x}^0 \succeq \boldsymbol{x}^1$, while if $\boldsymbol{x}^0 >> \boldsymbol{x}^1$ then $\boldsymbol{x}^0 \succ \boldsymbol{x}^1$*

Finally, we need an axiom to deal with the possibility of a non-convex region ruling out convex combinations of the $\sim(\boldsymbol{x}^0)$. We can deal with the problem by imposing a final assumption on preferences:

Axiom 2.6. ***Convexity.*** *$\boldsymbol{x}^1 \succeq \boldsymbol{x}^0 \implies t\boldsymbol{x}^1 + (1-t)\boldsymbol{x}^0 \succeq \boldsymbol{x}^0\ \forall\, t \in (0,1)$*

Axiom 2.7. ***Strict convexity.*** *$\boldsymbol{x}^1 \succeq \boldsymbol{x}^0 \wedge \boldsymbol{x}^1 \neq \boldsymbol{x}^0 \implies t\boldsymbol{x}^1 + (1-t)\boldsymbol{x}^0 \succ \boldsymbol{x}^0\ \forall\, t \in (0,1)$*

Any of the concavity axioms along with the previous set of axioms on preferences will roll out concave preferences and therefore, well behaved preferences that establish a consistent choice behavior. One of the most important implications of convexity is that it allows to establish a **marginal rate of substitution** by evaluating the absolute value of the slope of the indifference curve.

If preferences are strictly monotonic, any form of convexity requires the indifference curves to be at least weakly convex-shaped relative to the origin. This condition requires the marginal rate of substitution not to increase as the consumer moves from one bundle to the other. In other words, this set of conditions imply that the consumer is no more willing to give up one good for the other when the good shows high degrees of asymmetry in terms of its marginal rate of substitution.

In summary, we say that the axioms of completeness and transitivity describe a consumer that makes consistent comparisons amongst alternatives. The axiom of continuity is intended to guarantee the existence of well defined "at least as good" and "no better than" sets, and its mainly mathematical. All other axioms work to fully characterize a consumer and her taste over different objects of choice. Typically, it is required for her to display some form of non-satiation, either weak or strong, and some bias in favor of balancing consumption, either weak or strong.

2.2 The Utility Function

The utility function has no other purpose than being a vice to summarize the information contained in the consumer's preference relation:

Definition 2.5. Utility function. A real valued function $u : \mathbb{R}^n_+ \to \mathbb{R}$ is called a utility function representing the preference relation $\succeq$, if $\forall\, (\boldsymbol{x}^0, \boldsymbol{x}^1) \in \mathbb{R}^n_+,\ u(\boldsymbol{x}^0) \geq u(\boldsymbol{x}^1) \iff \boldsymbol{x}^0 \succeq \boldsymbol{x}^1$.

It is clear from this definition that if the utility function is to represent a preference relation, it must assign higher values to preferred bundles. In general, a binary relation that is complete, transitive and continuous can always be represented by a continuous real valued utility function. Nevertheless, we will impose more stringent conditions:

Theorem 2.1. ***Existence of a utility function.*** *If the binary relation $\succeq$ is complete, transitive, continuous and strictly monotonic, $\exists\, u : \mathbb{R}^n_+ \to \mathbb{R}$, which represents $\succeq$.*

It follows that existence does no imply uniqueness. In the case of utility functions, we can have an arbitrarily large number of functions that comply with the rules we have imposed on them. The only requirement is that they rank preferences in the same order as the preference relation. Moreover, we also have that transformations to utility functions can be made without altering such ordering properties:

Theorem 2.2. ***Invariance of the utility function to positive monotonic transformations.*** *Let $\succeq$ be a preference relation on $\mathbb{R}^n_+$ and suppose $u(\boldsymbol{x}^0)$ is a utility function that represents it. Then $v(\boldsymbol{x}^0)$ also represents $\succeq$ if and only if $v(\boldsymbol{x}^0) = f(u(\boldsymbol{x}^0))$ for every $\boldsymbol{x}^0$, where $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing on the set of values taken by u.*

As with the preference relation, we can have several useful properties that apply to properly defined utility functions:

Theorem 2.3. *Let* $\succeq$ *be represented by* $u : \mathbb{R}^n_+ \to \mathbb{R}$*. Then:*

1. $u(\boldsymbol{x})$ *is strictly increasing if and only if* $\succeq$ *is strictly monotonic,*

2. $u(\boldsymbol{x})$ *is quasiconcave if and only if* $\succeq$ *is convex,*[3]

3. $u(\boldsymbol{x})$ *is strictly quasiconcave if and only if* $\succeq$ *is strictly convex*

With the properties of utility functions defined, we can now introduce the proper vocabulary. The first order partial derivative, $\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{x}}$, is called the **marginal utility of good** i. As mentioned before, we can measure the degree with which the consumer substitutes between goods along the indifference set. Formally, the marginal rate of substitution of good j for good i ($MRS_{ij}(\boldsymbol{x})$) is given by the ratio of their respective marginal utilities:

$$MRS_{ij}(\boldsymbol{x}) = \frac{\partial u(\boldsymbol{x})/\partial x_i}{\partial u(\boldsymbol{x})/\partial x_j} \tag{1}$$

Whenever the marginal utilities are strictly positive the MRS_{ij} is also positive, and it can be interpreted as the rate at which a unit of good j can be exchanged for a unit of good i with no effect in the consumer's utility.

When $u(\boldsymbol{x})$ is continuously differentiable on $u : \mathbb{R}^n_{++}$, and preferences are strictly monotonic, the marginal utility of every good is virtually always positive, $\partial u(\boldsymbol{x})/\partial x_i) > 0$ for all bundles $\boldsymbol{x}$ and all $i = 1, ..., n$. When preferences are strictly convex, the marginal rate of substitution between two good is always strictly diminishing along any level of the surface of the function. In other words, for any quasiconcave utility function, its Hessian matrix, $\boldsymbol{H(x)}$, satisfies:

$$\boldsymbol{y}^T \boldsymbol{H(x)} \boldsymbol{y} \leq 0; \;\; \forall\, \boldsymbol{y} : \nabla u(\boldsymbol{x}) \cdot \boldsymbol{y} = 0$$

If this inequality is strict, then moving from $\boldsymbol{x}$ in any direction $\boldsymbol{y}$ that is tangent to the surface, will reduce utility.

3 The Consumer's Problem

When establishing the four building blocks required to model consumer choice, we have assumed that the consumer has a consumption set, $X = \mathbb{R}^n_+$, that contains all conceivable options in consumption. Her inclinations towards each of those combination are described by the preference relation, $\succeq$, defined on $\mathbb{R}^n_+$. The consumer's given circumstances can limit her ability to achieve certain combinations, and we acknowledged this limitation by defining a feasible set, $B \subset \mathbb{R}^n_+$. Finally, we will make the assumption that a consumer is motivated to choose the most preferred option in the feasible set according to her preference relation. Formally, she seeks:

$$\boldsymbol{x}^* \in B : \; \boldsymbol{x}^* \succeq \boldsymbol{x} \; \forall \, \boldsymbol{x} \in B$$

To deal with this problem, we will assume that the preference relation, $\succeq$, is complete, transitive, continuous, strictly monotonic and strictly convex on $\mathbb{R}^n_+$. Thus, it can be represented by a real-valued utility function, u, that is continuous, strictly increasing and strictly quasiconcave on $\mathbb{R}^n_+$.

[3] $f : D \to R$ is a concave function if $\forall\, \boldsymbol{x}^1, \boldsymbol{x}^2 \in D : f(\boldsymbol{x}^t) \geq t f(\boldsymbol{x}^1) + (1-t) f(\boldsymbol{x}^2) \; \forall\, t \in [0,1]$. $f : D \to R$ is a quasiconcave function if $\forall\, \boldsymbol{x}^1, \boldsymbol{x}^2 \in D : f(\boldsymbol{x}^t) \geq \min\{f(\boldsymbol{x}^1), f(\boldsymbol{x}^2)\} \; \forall\, t \in [0,1]$. The condition is strict if the relation holds without equality.

We say that the consumer operates within a **market economy**, where each good i has a strictly positive price $p_i > 0 \; \forall \; i = 1, ..., n$. We also assume that the size of the economy is big enough, so the choices of an individual consumer will not affect the price of any commodity. Therefore, the price vector $\boldsymbol{p} >> \mathbf{0}$ is fixed from the consumer's perspective.

The consumer is endowed with a fixed money income, $y \geq 0$, which she uses to purchase goods in the economy. More specifically, she will purchase x_i units of each good at price p_i, and therefore she spends $p_i x_i$ acquiring each good. It follows that the total expenditure cannot exceed the her income, $\sum_i p_i x_i \leq y \iff \boldsymbol{p} \cdot \boldsymbol{x} \leq y$. Formally, we can include all these assumptions in our definition of the feasible set, B, which now call the **budget set**.

$$B = \{\boldsymbol{x} | \boldsymbol{x} \in \mathbb{R}^n_+, \; \boldsymbol{p} \cdot \boldsymbol{x} \leq y\}$$

Therefore, the budget set gives us all possible bundles that can be afforded with a fixed level of income, y.

Now, we have all the tools required to solve the consumer problem in the market economy. With our assumptions we can write the consumer choice problem as the following **utility maximization problem**:

$$\max_{\boldsymbol{x} \in \mathbb{R}^n_+} u(\boldsymbol{x}), \quad \text{s.t.} \quad \boldsymbol{p} \cdot \boldsymbol{x} \leq y \tag{2}$$

If $\boldsymbol{x}^*$ solves this problem, we know that $\boldsymbol{x}^* \geq \boldsymbol{x} \; \forall \; \boldsymbol{x} \in B$, which means that $\boldsymbol{x}^* \succeq \boldsymbol{x} \; \forall \; \boldsymbol{x} \in B$. In line with our four building blocks assumptions, we can list several properties for this maximization problem:

1. The utility function $u(\boldsymbol{x})$ is real-valued and continuous,

2. The budget set is a non-empty ($\mathbf{0} \in \mathbb{R}^n_+$), closed, bounded ($\boldsymbol{p} > 0$), and thus compact set,[4]

3. By the Weirstrass existence theorem, we have that a maximum over $u(\boldsymbol{x}^*)$ exists,[5]

4. Since B is convex and the objective function is strictly concave, the maximizer of $u(\boldsymbol{x}^*)$ is unique,

5. Since preferences are strictly monotonic, the solution, $\boldsymbol{x}^*$, satisfies the budget constraint with equality lying on the boundary of the budget set. This condition implies that when $y > 0$, and because $\boldsymbol{x}^* \geq 0$ but $\boldsymbol{x}^* \neq 0$, $x_i^* \geq 0$ for at least one good i

The solution to this maximization problem can be written as function of the consumer's preferences, the price vector and her budget: $x_i^* = x_i(\boldsymbol{p}, y), \; i = 1, ..n$, or in vector notation $\boldsymbol{x}^* = \boldsymbol{x}(\boldsymbol{p}, y)$. When viewed as a function of $(\boldsymbol{p}, y)$, we denote the optimal quantities chosen by the consumer as **Marshallian (Walrasian) demand functions**.

We can now strengthen the requirements on the utility function to use calculus tools to explore demand behavior. If we now require $u(\boldsymbol{x})$ to be differentiable, we can solve the maximization problem using the the Lagrangian method:

$$\mathcal{L}(\boldsymbol{x}, \lambda) = u(\boldsymbol{x}) - \lambda[\boldsymbol{p} \cdot \boldsymbol{x} - y]$$

[4] A set S in $\mathbb{R}^n$ is called bounded if it is entirely contained within some ϵ-ball (either open or closed); that is S is bounded if $\exists \; \epsilon > 0 : S \subset B_\epsilon((x))$ for some $(x) \in \mathbb{R}$. A set S' is compact if S' is both closed and bounded.

[5] Let $f : s \to \mathbb{R}$ be a continuous real-valued mapping where S is a non-empty compact subset of $\mathbb{R}$. Then, $\exists \; \boldsymbol{x}^* \in S : f(\boldsymbol{x}') \leq f(\boldsymbol{x}) \leq f(\boldsymbol{x}^*) \; \forall \; \boldsymbol{x} \in S$.

We can apply the Kuhn-Tucker method to characterize the solution. If $\boldsymbol{x}^0 >> \mathbf{0}$ solves the maximization problem, then there exists a $\lambda^* \geq 0$ such that $(\boldsymbol{x}^*, \lambda^*)$[6] satisfies the following conditions:

$$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial u(\boldsymbol{x}^*)}{\partial x_i} - \lambda^* p_i = 0, \quad i = 1, ..., n,$$
$$\boldsymbol{p} \cdot \boldsymbol{x}^* - y \leq 0,$$
$$\lambda^*[\boldsymbol{p} \cdot \boldsymbol{x}^* - y] = 0$$

By strict monotonicity the budget constraint is satisfied with equality, so we can reduce the first order conditions to:

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial u(\boldsymbol{x}^*)}{\partial x_1} - \lambda^* p_1 = 0,$$
$$\vdots$$
$$\frac{\partial \mathcal{L}}{\partial x_n} = \frac{\partial u(\boldsymbol{x}^*)}{\partial x_n} - \lambda^* p_n = 0,$$
$$\boldsymbol{p} \cdot \boldsymbol{x}^* - y = 0$$

By strict monotonicity we have that $\partial u(\boldsymbol{x}^*)/\partial x_i > 0$ for some i. Since $\boldsymbol{p} >> \mathbf{0}$, the Lagrangian multiplier will be strictly positive at the solution because $\lambda^* = u_i(\boldsymbol{x}^*)/p_i > 0$. It follows that for all j, $\partial u(\boldsymbol{x}^*)/\partial x_j = \lambda^* p_j > 0$, so the marginal utility is proportional to the price for all goods at the optimum. If we combine these conditions we can conclude that at the optimum:

$$\frac{\partial u(\boldsymbol{x}^*)/\partial x_j}{\partial u(\boldsymbol{x}^*)/\partial x_k} = \frac{p_j}{p_k} \tag{3}$$

This expression indicates that the marginal utility of any two goods must be equal to the ratio of prices at the optimum. In other words, that the slope of the indifference curve (plane) must be equal to that of the budget constraint, and that the optimal quantity lies on the boundary, rather than in the interior of the budget set.

Theorem 3.1. ***Sufficiency of Consumer FOCs****. Suppose that $u(\boldsymbol{x})$ is continuous and quasi-concave on $\mathbb{R}^n_+$, and that $(\boldsymbol{p}, y) >> \mathbf{0}$. If u is differentiable at $\boldsymbol{x}^*$, and $(\boldsymbol{x}^*, \lambda^*) >> \mathbf{0}$ solves the consumer FOCs, then $\boldsymbol{x}^*$ solves the maximization problem at prices $\boldsymbol{p}$ and income y.*

This theorem implies that all we need to solve the consumer's problem is to find a solution $(\boldsymbol{x}^*, \lambda^*) >> \mathbf{0}$ for the FOCs of the maximization problem.

Until now we have made several assumptions to ensure that the Marshalian demand, $\boldsymbol{x}(\boldsymbol{p}, y)$ will be continuous on $\mathbb{R}^n_{++}$. Moreover, we would like to examine the slope of the demand curves in detail, and therefore we would like $\boldsymbol{x}(\boldsymbol{p}, y)$ to be differentiable:

[6] Let $f(\boldsymbol{x})$ and $g^j(\boldsymbol{x}),\ j = 1, ..., m$ be continuous real-valued functions defined over some domain $D \in \mathbb{R}^n$. Let $\boldsymbol{x}^*$ be an interior point of D and suppose that $\boldsymbol{x}^*$ maximises $f(\boldsymbol{x})$ on D subject to the constraints $g^j(\boldsymbol{x}) \leq 0,\ j = 1, ..., m$, and that f and each g^j are continuously differentiable on an open set containing $\boldsymbol{x}^*$. If the gradient vectors $\nabla g^j(\boldsymbol{x}^*)$ associated with constraints j that bind at $\boldsymbol{x}^*$ are linearly independent, then there is a unique vector $\lambda^* \in \mathbb{R}^n$, such that $(\boldsymbol{x}^*, \lambda^*)$ satisfies the Kuhn-tucker conditions:

$$\frac{\partial \mathcal{L}(\boldsymbol{x}^*, \lambda^*)}{\partial x_i} = \frac{\partial f(\boldsymbol{x}^*)}{\partial x_i} - \sum_j \lambda_j^* \frac{\partial g^j(\boldsymbol{x}^*)}{\partial x_i} = 0,\ i = 1, ..., n$$

$$\lambda_j^* \geq 0, \quad g^j(\boldsymbol{x}^*) \leq 0, \quad \lambda^* g^j(\boldsymbol{x}^*) = 0, \quad j = 1, ..., m$$

Theorem 3.2. *Let* $\boldsymbol{x}^* >> \mathbf{0}$ *solve the consumer's maximization problem at prices* $\boldsymbol{p}^0 >> \mathbf{0}$ *and income* $y^0 > 0$. *If*

1. *u is twice continuously differentiable on* $\mathbb{R}^n_{++}$,
2. $u(\boldsymbol{x}^*)/\partial x_i > 0$ *for some* $i = 1, ...n$, *and*
3. *The bordered Hessian of u has a non-zero determinant at* $\boldsymbol{x}^*$,

then, $\boldsymbol{x}(\boldsymbol{p}, y)$ *is differentiable at* $(\boldsymbol{p}^0, y^0)$.

4 The Indirect Utility Function

$u(\boldsymbol{x})$ is defined over a consumption set, X, and represents the consumer's preferences directly, hence it is called the **direct utility function**. Given prices $\boldsymbol{p}$ and income y, the consumer chooses the bundle $\boldsymbol{x}(\boldsymbol{p}, y)$ that maximizes the direct utility function. The relationship between prices, income and the level of maximized utility obtained can be expressed as:

$$v(\boldsymbol{p}, y) = \max_{\boldsymbol{x} \in \mathbb{R}^n_+} u(\boldsymbol{x}), \quad s.t.: \; \boldsymbol{p} \cdot \boldsymbol{y} \leq y \tag{4}$$

The function $v(\boldsymbol{p}, y)$ is known as the **indirect utility function**, and it is the maximum-value function corresponding to the consumer's utility maximization problem. In other words, the maximum level of utility that can be achieved as a function of prices and income that is obtained by choosing $\boldsymbol{x}(\boldsymbol{p}, y)$, would be given by :

$$v(\boldsymbol{p}, y) = u(\boldsymbol{x}(\boldsymbol{p}, y)) \tag{5}$$

It follows directly that the indirect utility function has several properties:

Theorem 4.1. *If* $u(\boldsymbol{x})$ *is continuos and strictly increasing on* $\mathbb{R}^n_+$, *then* $v(\boldsymbol{p}, y)$ *is:*

1. *Continuous on* $\mathbb{R}^n_{++} \times \mathbb{R}^n_+$,
2. *Homogenous of degree zero in* $(\boldsymbol{p}, y)$[7],
3. *Strictly increasing in* y,
4. *Decreasing in* $\boldsymbol{p}$,
5. *Quasiconcave in* $(\boldsymbol{p}, y)$, *and more importantly*
6. ***Roy's identity****: if* $v(\boldsymbol{p}, y)$ *is differentiable at* $(\boldsymbol{p}^0, y^0)$ *and* $\partial v(\boldsymbol{p}^0, y^0)/\partial y \neq 0$, *then*

$$x_i(\boldsymbol{p}^0, y^0) = -\frac{\partial v(\boldsymbol{p}^0, y^0)/\partial p_i}{\partial v(\boldsymbol{p}^0, y^0)/\partial y}, \quad i = 1, ..., n. \tag{6}$$

[7]A real-valued function $f(\boldsymbol{x})$ is called homogeneous of degree k if $f(t\boldsymbol{x}) \equiv t^k f(\boldsymbol{x}) \; \forall \; t > 0$.

5 The Expenditure Function

We have covered the consumer's problem in terms of choosing the bundle that maximizes her utility given a certain level of prices and income. We can also ask: what is the minimum level of money expenditure required to achieve a certain level of utility? The answer can be found by defining a level of expenditure as $e = \boldsymbol{p} \cdot \boldsymbol{x}$. It follows that e^* is the the minimum expenditure level that achieves utility u at prices $\boldsymbol{p}$. If the bundle that minimizes that expenditure is given by $\boldsymbol{x}^h$, then $e^* = e(\boldsymbol{p}, u) = \boldsymbol{p} \cdot \boldsymbol{x}^h$.

Formally, we define the **expenditure function** as a minimum-value function:

$$e(\boldsymbol{x}, u) = \min_{\boldsymbol{x} \in \mathbb{R}^n_+} \boldsymbol{p} \cdot \boldsymbol{x}, \quad \text{s.t.} \quad u(\boldsymbol{x}) \geq u \tag{7}$$

for all $\boldsymbol{p} >> \mathbf{0}$, where

$$e(\boldsymbol{p}, u) = \boldsymbol{p} \cdot \boldsymbol{x}^h \tag{8}$$

The quantities that satisfy this optimal set are a type of demand, just as the ones that solve that maximization problem. The difference is that these quantities are not directly observable. These are known as **Hicksian demands** and they reflect the effect of how a consumer substitutes goods when there are changes in prices in order to maintain a fixed level of utility.

The expenditure function also has some important properties. Let's denote the set of attainable utility levels as $U = \{u(\boldsymbol{x}) | \boldsymbol{x} \in \mathbb{R}^n_+\}$, then we have:

Theorem 5.1. *If $u(\bullet)$ is continuous and strictly increasing, then $e(\boldsymbol{p}, u)$ is:*

1. *Zero when u takes on the lowest level of utility in U,*
2. *Continuous on its domain $\mathbb{R}^n_{++} \times U$,*
3. *For all $\boldsymbol{p} >> \mathbf{0}$, strictly increasing and unbounded above in u,*
4. *Increasing in $\boldsymbol{p}$,*
5. *Homogenous of degree 1 in $\boldsymbol{p} >> \mathbf{0}$,*
6. *Concave in $\boldsymbol{p} >> \mathbf{0}$*

If, in addition, $u(\bullet)$ is strictly quasiconcave, we have

7. *Shephard's lemma: $e(\boldsymbol{p}, u)$ is differentiable in $\boldsymbol{p}$ at $(\boldsymbol{p}^0, u^0)$ with $\boldsymbol{p}^0 >> \mathbf{0}$ and*

$$\frac{\partial e(\boldsymbol{p}^0, u^0)}{\partial p_i} = \boldsymbol{x}_i^h(\boldsymbol{p}^0, u^0), \quad i = 1, ..., n \tag{9}$$

6 Relationship Between Indirect Utility and Expenditure

Despite representing different concepts, both the indirect utility and the expenditure functions are closely related by the underlying preferences of the consumer, a statement that can also be applied for Marshallian and Hicksian demands. In general, we have that:

Theorem 6.1. *Let $v(\boldsymbol{p}, y)$ and $e(\boldsymbol{p}, u)$ be the indirect utility and expenditure function for some consumer whose utility function is continuous and strictly increasing. Then for all $\boldsymbol{p} >> \mathbf{0}$, $y \geq 0$, and $u \in U$:*

1. $e(\boldsymbol{p}, v(\boldsymbol{p}, y)) = y$.

2. $v(\boldsymbol{p}, e(\boldsymbol{p}, y)) = u$.

The power of this theorem is that it allows to derive any of the consumer respective functions with only one optimization outcome. Going from one result to the other it's a matter of specifying the proper inverse function of each respective relationship. Moreover, we can also derive a direct relationship between quantities demanded:

Theorem 6.2. *If a consumer's preferences can be represented by a utility function that is continuous, strictly increasing, and strictly quasiconcave in* $\mathbb{R}^n_+$*, we have the following relationships between the Hicksian and Marshallian demand functions for* $\boldsymbol{p} >> 0$*,* $y \geq 0$*,* $u \in U$*, and* $i = 1, ..., n$*:*

1. $x_i(\boldsymbol{p}, y) = x_i^h(\boldsymbol{p}, v(\boldsymbol{p}, y))$

2. $x_i^h(\boldsymbol{p}, u) = x_i(\boldsymbol{p}, e(\boldsymbol{p}, u))$

In words, this theorem says that the Marshaling demand at prices $\boldsymbol{p}$ and income y is equal to the Hickcsian demand at prices $\boldsymbol{p}$ and the utility level that is the maximum that can be achieved at $(\boldsymbol{p}, y)$. On the other hand, Hicksian demand at prices $\boldsymbol{p}$ and utility u is the same as the Marshallian demand evaluated at those prices and an income level equal to the minimum expenditure necessary to achieve that utility level.

7 Properties of Consumer Demand

Until now we have made several assumptions about consumers that allowed us to model their behavior in a market economy. If our assumptions about preferences, objectives and circumstances are to be true, we should be able to predict the demand behavior in theory, and thus how it compares with what we observe in reality.

7.1 Relative prices and real income

When analyzing economic goods, it is ideal to take away the monetary component, as money is mostly a trading commodity. We define the **relative price** of a given good i as the number units that must be foregone of a good j to acquire one unit of the good in question. If p_i and p_j are the monetary prices of goods i and j, respectively. We define the relative price, $\bar{p}_{ij}$, as the ratio:

$$\bar{p}_{ij} = \frac{p_i}{p_j} \tag{10}$$

For real income, we use a similar notion to avoid the distortion of money and define real income as the potential of the consumer to acquire goods. We normalize this notion by examining her purchasing power over one good. If the consumer has a budget y, her real income, $\bar{y}_i$, is given by:

$$\bar{y}_i = \frac{y}{p_i} \tag{11}$$

If we revisit our result from utility maximization, we can see that only real prices and income matters for the consumer, as equal changes in monetary prices and income will leave the demand unchanged. This statement is the same as saying that the demand is homogenous of degree zero in prices and income.

Theorem 7.1. ***Homogeneity and budget balancedness.*** *If consumer preferences can be represented by a utility function that is continuous, strictly increasing, and strictly quasiconcave in* $\mathbb{R}^n_+$*, the consumer demand function* $x_i(\boldsymbol{p}, y),\ i = 1, .., n$*, is homogeneous of degree zero in all prices and income, and it satisfies budget balancedness,* $\boldsymbol{p} \cdot \boldsymbol{x}(\boldsymbol{p}, y) = y\ \forall\ (\boldsymbol{p}, y)$.

The homogeneity condition allows to effectively eliminate the monetary condition from the consumer problem. This is generally done by arbitrarily choosing one of the n goods as the numéraire to replace money. If its price is p_i, we can set $t = 1/p_i$, and using the homogeneity property we have:

$$\boldsymbol{x}(\boldsymbol{p}, y) = \boldsymbol{x}(t\boldsymbol{p}, ty) = \boldsymbol{x}\left(\left[\frac{p_1}{p_i}, ..., 1, ..., \frac{p_n}{p_i}\right], \frac{y}{p_i}\right) \tag{12}$$

This property entails that the demand for each of the n goods in the market, depends only on $n - 1$ relative prices and the consumer's real income.

7.2 Income and Substitution Effects

Another important property of the demand function is that we can ask questions about the consumer's behavior when changes occur in the economy. When the price of a given good declines, there might be two reasons why one should expect changes in the quantities demanded either of that good in specific, or for one or more of the other goods in the optimal bundle. 1) That good becomes relatively cheaper compared with the rest, and we should expect the consumer to substitute some of the relatively cheaper good quantity with some of the relatively more expensive ones; this is known as **substitution effect** (SE). 2) When the price of any good declines, the consumer's total command over all goods is effectively increased, allowing her to change purchases in the best way possible according to her preferences; this effect on quantity demanded as a result of the increased purchasing power is known as **income effect** (IE). Both of these effects will comprise the **total effect** (TE).

As we might expect, both Hicksian and Marshallian demands will provide us with this information. The relationship between the total, income and substitution effect can be summarized in what is known as the **Slutsky equation** or the **Fundamental Equation of Demand Theory**. Making the usual assumption about a well defined utility function we have:

Theorem 7.2. *Let* $\boldsymbol{x}(\boldsymbol{p}, y)$ *be the consumer's Marshaling demand system. Let* u^* *be the level of utility the consumer achieves at prices* $\boldsymbol{p}$ *and income* y*. Then,*

$$\frac{\partial x_i(\boldsymbol{p}, y)}{\partial p_j} = \frac{\partial x_i^h(\boldsymbol{p}, u^*)}{\partial p_j} - x_j(\boldsymbol{p}, y)\frac{\partial x_i(\boldsymbol{p}, y)}{\partial y}, \quad i, j = 1, .., n \tag{13}$$

with

$$TE = \frac{\partial x_i(\boldsymbol{p}, y)}{\partial p_j}$$
$$SE = \frac{\partial x_i^h(\boldsymbol{p}, u^*)}{\partial p_j}$$
$$IE = -x_j(\boldsymbol{p}, y)\frac{\partial x_i(\boldsymbol{p}, y)}{\partial y}$$

Despite not being obvious at first sight, the Slutsky equation will give us a close form tool to evaluate the behavior of the demand function. Consider the case when we want to see the effect of a price change of one good into its own demand. From our previous theorem we have:

$$\frac{\partial \boldsymbol{x}(\boldsymbol{p},y)}{\partial p_i} = \frac{\partial x_i^h(\boldsymbol{p},u^*)}{\partial p_i} - x_i(\boldsymbol{p},y)\frac{\partial x_i(\boldsymbol{p},y)}{\partial y} \tag{14}$$

The term on the right hand side is the slope of the Marshallian demand for good i. In other words, the response of the quantity demanded to a change on its own price. To describe that change, however, we need to know something about that first term on the right hand side. But thinking about it carefully, we already know that that term is the slope of the Hicksian demand. Despite being unobservable, we can use the Slutsky equation to link observable Mashalian demands with their substation terms. In order to do so, we can use this starting point to establish demand behavior:

Theorem 7.3. ***Negative own substitution terms***. *Let $x_i^h(\boldsymbol{p},u)$ be the Hicksian demand for goof i. Then*

$$\frac{\partial x_i^h(\boldsymbol{p},u)}{\partial p_i} \leq 0 \tag{15}$$

In words, that Hicksian demands are negatively slopped with respect to their own price. With these notions on our hands, we are now prepared to state the generally known as **law of demand**, where we can use some familiar terminology. A good is called **normal** if consumption increases as income increases, when holding prices constant. A good is called **inferior** if consumption of it declines as income increases, when holding prices constant.

Theorem 7.4. ***Law of Demand***. *A decrease in the own price of a normal good will cause quantity demanded to increase. If an own price decrease causes a decrease in quantity demanded, the good must be inferior.*

To see how substitution effects play a role in the law of demand, we can use what we know about Marshallian and Hicksian demands.

Theorem 7.5. ***Symmetric Substitution Terms***. *Let $x_i^h(\boldsymbol{p},u)$ be the consumer's system of Hicksian demands and suppose that $e(\bullet)$ is twice continuously differentiable. Then,*

$$\frac{\partial x_i^h(\boldsymbol{p},u)}{\partial p_j} = \frac{\partial x_j^h(\boldsymbol{p},u)}{\partial p_i}, \quad i,j = 1,...n \tag{16}$$

Furthermore, imagine we can arrange n^2 substitution terms in the consumer's entire demand system into an $n \times n$ matrix, with the own-substitution terms on the diagonal and the cross substitution terms off-diagonal. According to our previous theorems, all elements along the principal diagonal will be negative, and the matrix will be symmetric.[8] Moreover, this matrix must be negative semidefinite as well.[9]

[8] A matrix $\boldsymbol{M}$ is symmetric if $\boldsymbol{M} = \boldsymbol{M}'$. The entries of a symmetric matrix are symmetric with respect to the main diagonal, if the entries are written as $\boldsymbol{M} = (m_{ij})$, then $m_{ij} = m_{ji}\ \forall\ i,j$.

[9] A $n \times n$ matrix $\boldsymbol{M}$ is positive semidefinite (PSD)(negative semidefinite (NSD)) if $\forall\ z \in \mathbb{R}^n$,

$$z\boldsymbol{M}z \geq (\leq)0$$

If the inequality is strict for all $z \neq 0$, then $\boldsymbol{M}$ is positive definite (PD) (negative definite (ND))

Theorem 7.6. ***Negative Semidefinite Substitution Matrix.*** *Let $x_i^h(\boldsymbol{p}, u)$ be the consumer's system of Hicksian demands, and let*

$$\sigma(\boldsymbol{p}, u) = \begin{bmatrix} \frac{\partial x_1^h(\boldsymbol{p},u)}{\partial p_1} & \cdots & \frac{\partial x_1^h(\boldsymbol{p},u)}{\partial p_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n^h(\boldsymbol{p},u)}{\partial p_1} & \cdots & \frac{\partial x_n^h(\boldsymbol{p},u)}{\partial p_n} \end{bmatrix} \tag{17}$$

called the substitution matrix, contain all the Hicksian substitution terms. Then the $\sigma(\boldsymbol{p}, u)$ is negative semidefinite.

Now we can use the properties of the Hicksian demands to explore the main assertions of the law of demand. If we recall, the law states that changes in a good price will affect its demand in a certain way, depending on the nature of the good. With these new tools, we can now go even further and make claims about the entire system of substitution terms; we need not limit ourselves to statements about own-price and income changes. Using the substitution matrix, we can explore the effects of all price and income changes on the entire system of observable Marshallian demands.

Theorem 7.7. ***Symmetric and Negative Semidefinite Slutsky Matrix.*** *Let $x_i^h(\boldsymbol{p}, u)$ be the consumer's Marshallian demand system. Define the ij^{th} Slutsky term as*

$$\frac{\partial x_i(\boldsymbol{p}, y)}{\partial p_j} + x_j(\boldsymbol{p}, y)\frac{\partial x_i(\boldsymbol{p}, y)}{\partial y} \tag{18}$$

and form the entire $n \times n$ Slutsky matrix of price and income responses as follows:

$$\boldsymbol{s}(\boldsymbol{p}, y) = \begin{bmatrix} \frac{\partial x_1(\boldsymbol{p},y)}{\partial p_1} + x_1(\boldsymbol{p}, y)\frac{\partial x_1(\boldsymbol{p},y)}{\partial y} & \cdots & \frac{\partial x_1(\boldsymbol{p},y)}{\partial p_n} + x_n(\boldsymbol{p}, y)\frac{\partial x_1(\boldsymbol{p},y)}{\partial y} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n(\boldsymbol{p},y)}{\partial p_1} + x_j(\boldsymbol{p}, y)\frac{\partial x_1(\boldsymbol{p},y)}{\partial y} & \cdots & \frac{\partial x_n(\boldsymbol{p},y)}{\partial p_n} + x_n(\boldsymbol{p}, y)\frac{\partial x_n(\boldsymbol{p},y)}{\partial y} \end{bmatrix} \tag{19}$$

Then $\boldsymbol{s}(\boldsymbol{p}, y)$ is symmetric and negative semidefinite.

The requirements for consumer demand to satisfy homogeneity and budget balancedness, and that the associated Slutsky matrix be symmetric and negative semidefinite, provide a set of restrictions on allowable values for the parameters in any empirically estimated Marshallian demand system (as long as our assumptions seem realistic).

7.3 Elasticity Relations

To close the discussion on consumer demand, we examine the implications of the budget-balancedness condition on the consumer's choice subject to changes in price and income. From our previous discussion, recall that if $x_i(\boldsymbol{p}, u)$ is the consumer's Marshallian demand function, budget balancedness says that the budget constraint must hold with equality at every set of prices and income, or

$$y = \sum_i x_i(\boldsymbol{p}, u)p_i = \boldsymbol{x}(\boldsymbol{p}, u) \cdot \boldsymbol{p} \tag{20}$$

Since this condition holds with equality, we know that for any change in prices or income this equality must hold. Therefore, we need to be able to establish how the consumers adapt to a given change in order to maintain such equality. For this task, we make use of some useful definitions:

Definition 7.1. Let $x_i(\boldsymbol{p}, u)$ be the consumer's Marshallian demand for good i. Then let

$$\eta_{ij} \equiv \frac{\partial x_i(\boldsymbol{p}, u)}{\partial y} \frac{y}{x_i(\boldsymbol{p}, u)} \tag{21}$$

$$\epsilon_i \equiv \frac{\partial x_i(\boldsymbol{p}, u)}{\partial p_j} \frac{p_j}{x_i(\boldsymbol{p}, u)} \tag{22}$$

and let

$$s_i \equiv \frac{p_i x_i(\boldsymbol{p}, u)}{y}, \quad \text{so that} \quad \sum_i s_i = 1 \tag{23}$$

We call η_i the **income elasticity** of demand for good i, and it measures the percentage change in the quantity of i demanded per one percent change in income. ϵ_{ij} is called the **price elasticity** of demand for good i, and it measures the percentages change on the quantity of i demanded per one percent change in the price of good j. Whenever $i = j$, ϵ_{ij} is called **own-price elasticity** of demand for good i, while if $i \neq j$ ϵ_{ij} is called **cross-price elasticity** of demand for good i with respect to p_j. Finally s_i denotes the **income share**, or the proportion of the consumer's income that is spent purchasing good i.

Theorem 7.8. ***Aggregation in Consumer Demand****. Let $\boldsymbol{x}(\boldsymbol{p}, u)$ be the consumer?s Marshallian demand system. Then the following relations must hold among income shares, price, and income elasticities of demand:*

1. *Engel aggregation:* $\sum_i s_i \eta_i = 1$
2. *Cournot aggregation:* $\sum_i s_i \epsilon_{ij} = s_j, \quad j = 1, ..., n$

Putting together Theorems 10 through 17, we have set the ground for the logical implications of utility maximizing behavior. Homogeneity provides insights about how demand responds to an overall, equiproportional change in all prices and income simultaneously, and budget balancedness requires that demand always exhaust consumer income. The Slutsky equation provides qualitative information on how demand responds to general price changes, as well as analytically uncovering properties of the unobservable components of the demand behavior subject to price changes: income and substitution effects. Finally, aggregations allow us to examine how quantities demanded have to adapt across the system of demand functions.

8 Revealed Preference

In the previous sections we have analyzed demand by assuming the consumer exhibits certain characteristics in her preferences, (complete, transitive, and strictly monotonic), and subsequently we derived analytically how we expect her to behave in a market economy, in a way that we can observe and analyze. This type of analysis, however, relies on our assumptions on preferences and how we contrast those with what we can observe in the real world economy.

We can, however, make assessments on consumer behavior based solely on observations and without having to make assumptions *ex-ante* about preferences. The idea is that if a consumer buys one bundle when she could buy another one, that bundle is said to be **revealed preferred** to the second. Instead of imposing axioms on a consumer's tastes, we make assumptions about the consistency of her choices. Formally we say that:

Definition 8.1. Weak Axiom of Revealed Preference (WARP). A consumer's choice behavior satisfies WARP if for every distinct pair of bundles $\boldsymbol{x}^0$, $\boldsymbol{x}^1$ with $\boldsymbol{x}^0$ chosen at prices $\boldsymbol{p}^0$ and $\boldsymbol{x}^1$ chosen at prices $\boldsymbol{x}^1$,

$$\boldsymbol{p}^0\boldsymbol{x}^1 \leq \boldsymbol{p}^0\boldsymbol{x}^0 \implies \boldsymbol{p}^1\boldsymbol{x}^0 > \boldsymbol{p}^1\boldsymbol{x}^1 \tag{24}$$

In other other words, whenever $\boldsymbol{x}^0$ is revealed preferred to $\boldsymbol{x}^1$, $\boldsymbol{x}^1$ is never revealed preferred to $\boldsymbol{x}^0$.

The most intuitive way to understand this axiom is to think about how a consumer can show consistency in the bundles she chooses given a certain budget, y, and combination of prices, $\boldsymbol{p}$. The axiom says that $\boldsymbol{x}^0$ is revealed preferred to $\boldsymbol{x}^1$ whenever the consumer chooses $\boldsymbol{x}^0$ over $\boldsymbol{x}^1$, whenever her budget allows her to choose between the two. She takes $\boldsymbol{x}^1$ if and only if she cannot afford $\boldsymbol{x}^0$.

In addition to **WARP**, we have to make one more assumption about the consumer's preferences. Let $\boldsymbol{x}(\boldsymbol{p}, y)$ be a choice function of the consumer (not a demand function). We say that for $\boldsymbol{p} >> 0$, the choice $\boldsymbol{x}(\boldsymbol{p}, y)$ satisfies budget balancedness, $\boldsymbol{p} \cdot \boldsymbol{x}(\boldsymbol{x}, y) = y$. With the combination of these two requirements on consumers' preferences, we get a set of remarkable implications.

Theorem 8.1. *Revealed preferences and WARP. If a consumer satisfies WARP and budget balancedness, her choice function, $\boldsymbol{x}(\boldsymbol{p}, y)$, satisfies the following properties:*

1. *$\boldsymbol{x}(\boldsymbol{p}, y)$ is homogenous of degree zero in $(\boldsymbol{p}, y)$,*
2. *the Slutsky matrix of the choice function is negative semidefinite,*
3. *for two goods, the Slutsky matrix is symmetric*

From this theorem, it is worth noting that if the choice function $\boldsymbol{x}(\boldsymbol{p}, y)$ happens to be a utility-based demand function, the it must satisfy WARP. It is tempting suggest that this relation holds backwards, but that is not the case for more than two goods. For more than two goods, WARP and budget balancedness imply neither symmetry of the Slutsky Matrix nor the absence of intransitive cycles in the revealed preferred to relation.

This problem leads to the question, how can we strengthen WARP to get a theory of revealed preference that is equivalent to the theory of utility maximization? That's why we need the following definition:

Definition 8.2. Strong Axiom of Revealed Preference (SARP). A consumer's choice behavior satisfies **SARP** if for every sequence of bundles, $\boldsymbol{x}^0, \boldsymbol{x}^1, ..., \boldsymbol{x}^k$, where $\boldsymbol{x}^0$ is revealed preferred to $\boldsymbol{x}^1$, and $\boldsymbol{x}^1$ is revealed preferred to to $\boldsymbol{x}^2$, ..., and $\boldsymbol{x}^{k-1}$ is revealed preferred to $\boldsymbol{x}^k$, it is not the case that $\boldsymbol{x}^k$ is revealed preferred to to $\boldsymbol{x}^1$.

With SARP we can now rule out any intransitive revealed preferences and therefore it can be used to induce a complete and transitive preference relation $\succeq$, for which will exist a utility function that rationalizes that behavior.

To conclude this section we introduce one more axiom for revealed preferences that allows us to examine revealed preferences without observing the entire set of bundles, as SARP requires:

Definition 8.3. Generalized Axiom of Revealed Preferences (GARP). A finite set of observed price and quantity data satisfies **GARP** if and only if there exists a continuous increasing, and concave utility function that rationalizes the data.

Although powerful, it is difficult to establish that observed data does not violate GARP. The challenge is now to establish the acceptable conditions that allow to ignore minor violations of GARP, and how algorithms are able to ignore such violation to efficiently derive the utility functions that rationalize the observed data.

9 Uncertainty

So far, our study of consumers and their choice behavior has been inserted in a deterministic world. This condition implies that the consumer knows the prices of all commodities and knows that any particular bundle can be obtained with certainty. In the real world, however, the consumer is inserted in a reality of uncertainty where prices, accessibility, and income are subject to some degree of uncertainty. It is necessary then, to evaluate how the consumer might behave in the presence of uncertainty.

9.1 Preferences

When dealing with uncertainty we have to maintain our assumptions on preference relations, but instead of consumption bundles we will work with preference relations over **gambles**. Let $A = \{a_1, ..., a_n\}$ denote the finite set of **outcomes**. a_i can denote consumption bundles, amount of money (positive or negative), or anything at all. The idea is that a_i involves no uncertainty. Nevertheless, we use the set A as the basis for constructing gambles.

A **simple gamble** assigns a probability p_i, to each outcome $a_i \in A$. The basic properties of probabilities hold, and we say that $p_i \geq 0$, and $\sum_i p_i = 1$. We denote a simple gamble as $(p_1 \circ a_1, ..., p_n \circ_n)$. Formally:

Definition 9.1. Simple Gambles. Let $A = \{a_1, ..., a_n\}$ be the set of outcomes. Then $\mathcal{G}_\mathcal{S}$, the set of simple gambles (on A), is given by

$$\mathcal{G}_\mathcal{S} \equiv \left\{ (p_1 \circ a_1, ..., p_n \circ_n) | p_i \geq 0, \sum_{i=1}^{n} = 1 \right\} \tag{25}$$

It follows that not all gambles are simple. The prime example is lotteries that provide as prize, a ticket for the next lottery. These are called **compound gambles**. It is obvious that there is no limit to the level of computing that such gambles involve. To simplify our treatment then, we will limit our gambles to leave out any infinitely compound gambles, and require that all gambles must result in an outcome in A.

We denote the set of all gambles, simple and compound, as $\mathcal{G}$. For any gamble $g \in \mathcal{G}$, then $g = (p_1 \circ g^1, ..., p_k \circ g^k)$, for some $k \geq 1$ and some gambles $g^i \in \mathcal{G}$, where g^i might be compound gambles, simple gambles or outcomes.

The objects of choice under uncertainty are gambles. Analogous to the consumer theory case, we can assume that the decision maker has preferences, $\succeq$, over the set of gambles, $\mathcal{G}$. As before, we need to impose some axioms on the decision making process, called **axioms of choice under uncertainty** for the preference relation $\succeq$. We let $\sim$ and $\succ$ denote "indifferent to" and "strictly preferred to" relations induced by $\succeq$. We have:

Axiom 9.1. ***Completeness***. *For any two gambles g and g' in $\mathcal{G}$, either $g \succeq g'$ or $g' \succeq g$.*

Axiom 9.2. ***Transitivity***. *For any three gambles g, g' and g'' in $\mathcal{G}$, if $g \succeq g'$ and $g' \succeq g''$, then $g \succeq g''$.*

Since each outcome $a_i \in A$ is represented in $\mathcal{G}$ as a degenerate gamble, axioms (9.1) and (9.2) imply that the finitely many elements of A are ordered by $\succeq$. So we can further say w.l.o.g that the elements in A are indexed as $a_1 \succeq a_2 \succeq ... \succeq a_n$.

From this indexing process, it would be fair to say that no gamble is better than giving a_1 with certainty, and that no gamble is worse than giving a_n with certainty. In other words, for any gamble g, $(\alpha \circ a_1, (1-\alpha) \circ a_n) \succeq g$ when $\alpha = 1$; and $g \succeq (\alpha \circ a_1, (1-\alpha) \circ a_n)$ when $\alpha = 0$. So we can establish our next axiom on continuity:

Axiom 9.3. ***Continuity***. *For any gamble g in $\mathcal{G}$, there is some probability, $\alpha \in [0,1]$, such that $g \sim (\alpha \circ a_1, (1-\alpha) \circ a_n)$.*

Axiom (9.3) implies that there is always a gamble that makes the consumer indifferent between getting the price or a linear combination (expectation) between the most and least preferred outcomes in the outcome set. It seems natural to now deal with preferences between gambles:

Axiom 9.4. ***Monotonicity***. *For all probabilities $\alpha, \beta \in [0,1]$,*

$$(\alpha \circ a_1, (1-\alpha) \circ a_n) \succeq (\beta \circ a_1, (1-\beta) \circ a_n) \iff \alpha \geq \beta \tag{26}$$

With this axiom, we say that $a_1 \succ a_n$, which rules out the possibility of the consumer being indifferent across the entire outcome set. Next, we have to consider the possibility of a consumer begin indifferent across gambles:

Axiom 9.5. ***Substitution***. *If $g = (p_1 \circ g^1, ..., p_k \circ g^k)$, and $h = (p_1 \circ h^1, ..., p_k \circ h^k)$ are in $\mathcal{G}$, and if $h^i \sim g^i \; \forall \, i \implies h \sim g$*

Together with Axiom (9.1), Axiom (9.5) says that whenever an agent is indifferent between two gambles, he must also be indifferent across all linear combinations between those two gambles. That is, if $g \sim h$, then by Axiom (9.1) $g \sim g$. Axiom (9.5) implies $(\alpha \circ g, (1-\alpha) \circ h) \sim (\alpha \circ h, (1-\alpha) \circ g) = g$.

We only need one more axiom to characterize the consumer's choice under uncertainty. In particular, we need to establish a rule on how, when a consumer considers a gamble, she should care only about the effective probabilities that the gamble assigns to each outcome in A. To understand this concept consider $A = \{a_1, a_2\}$, and the compound gamble yielding a_1 with probability α, and yielding a lottery ticket with probability $(1-\alpha)$. The lottery itself is a simple gamble that yields a_1 with probability β and a_2 with probability $(1-\beta)$. The probability of getting outcome a_1 has to consider both the probability of getting a_1 the first time, or with the lottery. That is, the probability of a_1 is $\alpha + (1-\alpha)\beta$, while the probability of getting a_2 is $(1-\alpha)(1-\beta)$. In this little example, to say that the decision maker cares only about the effective probabilities, is equivalent to say that the consumer is indifferent between the compound gamble or the simple gamble $(\alpha + (1-\alpha)\beta \circ a_1, (1-\alpha)(1-\beta) \circ a_2)$ that it induces.

Formally, we say that for any gamble $g \in \mathcal{G}$, if p_i denotes the probability assigned to a_i by g, then g **induces the simple gamble** $(p_1 \circ a_1, ..., p_n \circ a_n) \in \mathcal{G}_{\mathcal{S}}$. Every $g \in \mathcal{G}$ induces a **unique** simple gamble:

Axiom 9.6. ***Reduction to simple games***. *For any gamble $g \in \mathcal{G}$, if $(p_1 \circ a_1, ..., p_n a_n)$ is the simple gamble induced by g, then for $(p_1 \circ a_1, ..., p_n a_n) \sim g$.*

Note that Axiom (9.6) together with Axiom (9.2) will lead to individual preferences over all gambles (simple and compound) to be completely determined by his preferences over simple gambles.

9.2 Von Neumann-Morgenstern Utility

Following the same steps as we did for choices under certainty, we can start describing the decision making process by an appropriate utility function. Suppose that $u : \mathcal{G} \to \mathbb{R}$ is a utility function representing $\succeq$ on $\mathcal{G}$. So, for every $g \in \mathcal{G}$, $u(g)$ denotes the utility number assigned to gamble g. For every i, u assigns the number $u(a_i)$ to the degenerate gamble $(1 \circ a)$, in which outcome a_i occurs with certainty.

Definition 9.2. Expected Utility Property. The utility function $u : \mathcal{G} \to \mathbb{R}$ has the expected utility property if, for every $g \in \mathcal{G}$,

$$u(g) = \sum_{i=1}^{n} p_i u(a_i) \tag{27}$$

with $(p_1 \circ a_1, ..., p_n \circ a_n)$ as the simple gamble induced by g.

That is, the utility of the gamble is equal to the sum of effective probabilities of g yielding $u(a_i)$. It follows that if $g_s = (p_1 \circ a_1, ..., p_n \circ a_n)$ is a simple gamble, then because the simple gamble implied by g_s is g_s itself, it is true that:

$$u(p_1 \circ a_1, ..., p_n \circ a_n) = \sum_{i=1}^{n} p_i u(a_i), \quad \forall\ (p_1, ..., p_n) \tag{28}$$

The function u, then, is completely determined on all of $\mathcal{G}$ by the values it assumes on the finite set of outcomes, A.

If an individual's preferences are represented by a utility function with the expected utility property, and if that person always chooses his most preferred alternative available, then that individual will choose one gamble over another if and only if the expected utility of the one exceeds that of the other. Consequently, such an individual is an **expected utility maximiser**.

It becomes clear that the expected utility function assumption imposes a great degree of limitations on preferences, so they can be expressed as linear combinations of independent utilities. To maintain a difference between such functions, we define utility functions possessing the expected utility property as **von Neumann-Morgenstern (VNM) utility functions**.

Theorem 9.1. ***Existence of a VNM Utility Function on*** $\mathcal{G}$. *Let preferences $\succeq$ over gambles in $\mathcal{G}$ satisfy Axioms (9.1)-(9.6). Then there exists a utility function $u : \mathcal{G} \to \mathbb{R}$ representing $\succeq$ on $\mathcal{G}$, such that u has the expected utility property.*

The VNM utility function allows to have a mapping between the preferences of an individual and a given set of gambles. For instance, an individual is indifferent between gambles if her VNM utility has the same value for both. Nevertheless, and unlike the consumer theory we specified before, the VNM utility is not only an ordinal instrument, and thus it cannot be manipulated as regular utility functions. Consider $A = \{a, b, c\}$, where $a \succ b \succ c$, and that $\succeq$ satisfies Axioms (9.1)-(9.6). We know by Axioms (9.3) and (9.4) that $\exists\ \alpha \in (0, 1)$ satisfying:

$$b \sim (\alpha \circ a, (1 - \alpha) \circ c) \tag{29}$$

If we let u be some VNM representation of $\succeq$. Then the preceding indifference relation implies:

$$\begin{aligned} u(b) &= u(\alpha \circ a, (1 - \alpha) \circ c) \\ &= \alpha u(a) + (1 - \alpha) u(c) \qquad \text{(By Expected Utility Property)} \end{aligned} \tag{30}$$

We can rearrange this equality to:

$$\frac{u(a) - u(b)}{u(b) - u(c)} = \frac{1 - \alpha}{\alpha} \tag{31}$$

Therefore, the ratios of the differences between the preceding utility values are determined by α. Since α is uniquely determined by the preferences of the decision maker, so is the preceding ratio of utility differences. In other words, VNM utility representations provide more than just an ordinal indication, and therefore monotone transformations might not yield another VNM utility representation with the expected utility property. Nevertheless, we still have an analogous property:

Theorem 9.2. ***VNM Utility Functions are Unique up to Positive Affine Transformations***. *Suppose that the VNM utility function $u(\bullet)$ represents $\succeq$. Then the VNM utility function, $v(\bullet)$, represents those same preferences if and only if for some scalar α and some scalar $\beta > 0$,*

$$v(g) = \alpha + \beta u(g) \tag{32}$$

for all gambles g.

So far, with the assumptions we have set on the agent's binary comparisons between gambles in the underlying preference relation, we still cannot use VNM utility functions for interpersonal comparisons of well-being, nor can we measure the "intensity" with which one gamble is preferred to another.

9.3 Risk Aversion

To establish a measure for how a decision maker reacts to risk, we focus on gambles that take wealth as the set of outcomes, moreover, a positive one $A = \mathbb{R}_+$. A simple gamble now takes the form $(p_1 \circ w_1, ..., p_n, w_n)$, where n is some positive integer, the w_i's are non-negative wealth levels, and the non-negative probabilities, $p_1, ..., p_n$ add up to one. We further assume that the individual's VNM utility function, $u(\bullet)$, is differentiable with $u'(w) > 0$ for all levels of wealth w.

The expected value of a simple gamble g, offering w_i with probability p_i, is given by $E[g] = \sum_{i=1}^{n} p_i w_i$. Suppose the agent is given a choice between accepting a gamble g, or receiving with **certainty** the expected value of g. If $u(\bullet)$ is the agent's VNM utility function, we can evaluate these two alternatives:

$$u(g) = \sum_{i=1}^{n} p_i u(w_1) \tag{33}$$

$$u\left(\mathbb{E}[g]\right) = u\left(\sum_{i=1}^{n} p_i w_i\right) \tag{34}$$

Equation (33) is known as the VNM utility of the gamble, while the second is the VNM utility of the gamble's expected value. According to the axioms of decision making under uncertainty, an agent will prefer the option with the highest expected utility. If someone would rather receive the expected value of the gamble with certainty than face the gamble's risk, the agent is said to be **risk averse**. If the agent would take the bet rather than the expected utility with certainty, the agent is said to be **risk loving**.

We can focus on the characteristics of u on $\mathcal{G}_\mathcal{S}$ to capture an individual's attitude towards risk:

Definition 9.3. Risk Aversion, Risk Neutrality, and Risk Loving. Let $u(\bullet)$ be an individual's VNM utility function for gambles over non-negative levels of wealth. Then for the simple gamble $g = (p_1 \circ w_1, ..., p_n \circ w_n)$, the individual is said to be:

1. risk averse at g if $u(\mathbb{E}[g]) > u(g)$,
2. risk neutral at g if $u(\mathbb{E}[g]) = u(g)$,
3. risk loving at g if $u(\mathbb{E}[g]) < u(g)$

Each of these attitudes towards risk can be linked to a particular property of the VNM, so we can also say:

1. risk averse if the VNM utility function is strictly concave in wealth,
2. risk averse if the VNM utility function is linear in wealth,
3. risk averse if the VNM utility function is strictly convex in wealth

When an individual faces the choice of taking a gamble, there would be some amount of wealth offered with certainty that'd make her indifferent between taking the gamble or that wealth with certainty. We call that amount the **certainty equivalent**. If a person is risk averse and strictly prefers more money to less, they would value the certainty amount more that the expected value of the gamble, and they would be willing to pay in order to avoid the gamble's risk:

Definition 9.4. Certainty Equivalent and Risk Premium. The certainty equivalent of any simple gamble, g, over wealth level is an amount of wealth, CE, offered with certainty, such that $u(g) \equiv u(CE)$. The risk premium is an amount of wealth, P, such that $u(g) \equiv u(\mathbb{E}[g] - P)$. Clearly, $P \equiv \mathbb{E}[g] - CE$.

When making decision under uncertainty, we are not only concerned with establishing that if an individual is risk averse or not, we would also like to have a measure of risk aversion. To do so, we use the **Arrow-Pratt measure of risk aversion**.

Definition 9.5. The Arrow-Pratt measure of absolute risk aversion is given by:

$$R_a(w) = \frac{-u''(w)}{u'(w)} \tag{35}$$

The sign of this indicator tells the individual's attitude towards risk: $R_a(w)$ is positive, negative, or zero if the agent is risk averse, risk loving, or risk neutral, respectively. Moreover, any affine transformation will leave this measure unchanged. Nonetheless, $R_a(w)$ is only a local measure of risk aversion, so it need not to be the same at every level of wealth. For example, one could expect that attitudes towards risk vary according to the level of wealth.

We can say that a VNM utility function displays **constant (CARA), decreasing (DARA), or increasing (IARA)** absolute risk aversion over some domain of wealth if $R_a(w)$ is constant, decreasing, or increasing in wealth over a given wealth domain, respectively. We can refine our measure of risk aversion by including a the notion of **relative risk aversion**:

Definition 9.6. The Arrow-Pratt measure of relative risk aversion is given by:

$$R_r(w) = w\frac{-u''(w)}{u'(w)} \tag{36}$$

Similarly, we can observe how the degree of risk aversion relates to given levels of wealth. We say that a VNM utility function displays **constant (CRRA), decreasing (DRRA), or increasing (IRRA)** relative risk aversion over some domain of wealth if $R_r(w)$ is constant, decreasing, or decreasing over a given wealth domain, respectively.